From Quantum Oscillators to Landau-Fock-Darwin model: A Statistical Thermodynamical Study

Jishad Kumar* and Ebad Kamil
Indian Institute of Science Education and Research-Kolkata, Mohanpur Campus, Nadia - 741252
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We analyse the thermodynamic quantities of some simple oscillator systems like bosonic harmonic oscillator, fermionic harmonic oscillator and a supersymmetric harmonic oscillator (which is a combination of a bosonic and a fermionic oscillators), and discuss in detail about the nature of the specific heat, internal energy and entropy of these systems both at low and high temperatures. Also we have studied the behavior of the thermodynamic properties of the well known Landau-Fock-Darwin (LFD) problem viz., an electron in the combined presence of a uniform magnetic field acting perpendicular to the plane of motion of the electron and an isotropic and cylindrically symmetric parabolic potential well in the directions normal to the field with a natural confinement frequency ω_0. We have succinctly derived the respective thermodynamic quantities of LFD model with and without the intrinsic spin angular momentum of the electron. For the sake of simplicity and convenience we have taken the simple case of an electron carrying spin $\frac{1}{2}$.

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I. INTRODUCTION

Thermodynamics; the phenomenological branch of physics, is a summary of the properties that real physical systems (in thermal equilibrium) exhibit. Thermodynamics was established in the 19th century by a group of pioneers with the formulation of the law of conservation of energy relating the equivalence of work and heat as stated by Robert Mayer and Hermann Helmholtz[1]. The total dynamics of a system can be defined independantly in terms of the main constituents of thermodynamics like entropy, temperature and the three laws connecting these state variables. This could be a reasonable argument for the robustness of thermodynamics over the last century, eventhough some of the profound peers like classical mechanics and electrodynamics underwent a lot of structural changes.

Thermodynamics shows its full glory when it couples with Statistical physics, particularly for a microsopic system. Statistical physics is that part of physics which derives emergent properties of macroscopic matter from the atomic structure and the microscopic dynamics. Emergent properties here means those properties like temperature, perssure, dielectric and magnetic constants etc, which are essentially determined by the interaction of many particles (atoms or molecules). Exclusively, these emergent propeties are typical for many-body systems and they do not exist (in general) for microscopic systems. Strictly speaking, statistical physics is the bridge between the microscopic and the macroscopic world and it provides methods for calculating the macroscopic properties (for eg., like the specific heat), from the microscopic information(like the interaction energy between the particles). Statistical mechanics introduces probabilities into physics and connects them with the fundamental physical quantity *'entropy'* (The name *'entropy'* was given after the work of Clausius in 1865 [2] and also James Clerk Maxwell [3] in 1878 through his "*Gedankenexperiment*". In fact, Maxwell proposed the name "Statistical Mechanics".). The coalescence of thermodynamics and statistical physics led to the emergence of Statistical thermodynamics, established through the well known Boltzmann-Planck [4] formula for the Clausius entropy $S = k_B \ln W$ (W is the thermodynamic probability) and the ergodic hypothesis by Boltzmann [5], also independantly by Josiah Williard Gibbs [6], the father of the ensemble theory. In classical statistical mechanics we did not know the exact microscopic state of the system (location in the phase space). We made use of the ergodic hypothesis and replaced the time average of a physical quantity with an ensemble average ie., an average over many equivalent systems.

All these developments in classical statistical mechanics almost coincided with the revolutionary concept of quantum theory by Max Planck, that changed the entire structure of physics. It would have been surprising if statistical mechanics could have escaped the repercussions of the quantum revolution. But the whole structure of statistical mechanics was overhauled by the introduction of the concept of indistinguishability of (identical) particles. But the concept of ergodicity remains untouched in the quantum case also. Microcanonical and canonical ensembles retained their positions with some modifications and grand canonical ensemble made a debut. The statistical aspect which already exists in classical statistical mechanics in view of the large number of particles present in the system, has been augmented by the statistical aspect came from the probabilistic nature of the wave mechanical description. All these new concepts reformulated the ensemble theory with the introduction of density matrix, which is the quantum mechanical analogue of the density function of the classical phase space,

*Electronic address: jkumar@iiserkol.ac.in

first introduced by Landau [7] and later by von Neumann [8], later on more rigorously by Dirac [9].

With these remarks we are led to analyze the quantum thermodynamic properties of some simple oscillator systems like a bosonic harmonic oscillator, a fermionic harmonic oscillator and finally a supersymmetric harmonic oscillator (which combines a bosonic and a fermionic oscillator). Harmonic oscillator is the model system of model systems and it is effectively a single particle problem, ie., it is a problem where the particle is bounded within a quadratic potential. Hence it is always intutive to study the basic quantum thermodynamical properties of a particle trapped in a quadratic potential irrespective of the inherent character of the particle (Boson or Fermion). Classically speaking, the harmonic oscillator system is like a particle moving in the x-direction connected by a spring to a fixed point. Its potential energy is $V(x) = \frac{kx^2}{2}$, where k is the force constant of the spring attached to the particle. In the case of a classical oscillator, the amplitude A of the periodic motion can take a continuum of values, and hence the energy $E = \frac{1}{2}m\omega^2 A^2$, starting from $E = 0$. But quantum mechanical harmonic oscillator can only take certain discrete energy values since the energy eigen spectrum is fully quantized or discretized as we will see in the forthcoming sections in detail.

Our main aim is to study the statistical thermodynamical quantities like internal energy, entropy and specific heat (an attribute of the third law of thermodynamics [10]). Third law says that the specific heat and the limiting entropy of a quantum system vanishes as the equilibrium temperature is approached. Perhaps Einstein's work in 1907 [11](considered as the origin of quantum statistics) proposed that quantum effects lead to the vanishing of the specific heat at zero temperature. In addition to that, we consider the effect of magnetic field on the thermodynamic properties of an electron in the combined presence of a uniform magnetic field and a parabolic potential.

The quantum mechanical problem of an electron in a magnetic field was first solved by Landau[12]. Peierls[13] elucidated the concept of diamagnetism of free electrons under the influence of a magnetic field, followed by Darwin[14] who explained the role of boundary for the recovery of correct diamagnetism. Even from the classical mechanical point of view this problem was quite interesting and surprising. The famous van Leeuwen[15] theorem was a fatal blow to some of the older theories of diamagnetism that abounds with subtle pitfals because of the reason that the correct bulk susceptibility calculation needed a boundary effect. Later on the problem of an electron in a magnetic field has been immensively exploited and has got innumerable attention in terms of quantum transport phenomena like quantum hall effect[16].

Another paradigm, an electron in the combined presence of a uniform magnetic field and a parabolic potential was comprehensively solved by Fock[17] (nowonwards we call this model as Landau-Fock-Darwin (LFD) model). More recently LFD model has been extensively studied under the influence of a dissipative quantum heat bath of non-interacting oscillators for the calculation of dissipative Landau diamagnetism[18] and to verify the third law of thermodynamics[19]. In this paper we are not interested in the much studied Landau diamagnetism, but we will be looking at the equilibrium thermodynamics of the LFD problem.

We restrict ourselves to equilibrium statistical mechanics, which aims at characterising the equilibrium state in terms of given constraints, without regard to the mechanism relating to how this equilibrium state is brought about. Since the systems we are interested in are those with the number of particles fixed, we can use canonical ensemble formalism to obtain the thermodynamic quantities.

With the preceding introduction the organization of the paper is as follows: In Sec.2, we review the well known thermodynamic quantities which we want to look at. In the following subsection, we revisit the quantum thermodynamics of a simple bosonic harmonic oscillator. Other two subsections mainly decribe the thermodynamics of a fermionic harmonic oscillator and a supersymmetric harmonic oscillator. In Sec.3, we analyse the problem of an electron in the combined presence of a magnetic field and a parabolic confinement potential. Here we discuss the two cases viz., electron with and without its inherent spin. In Sec.4, we discuss the results and finally, Sec.5 is devoted to the concluding remarks.

II. STATISTICAL THERMODYNAMIC QUANTITIES REVISITED

Canonical ensemble description is the most useful one, as in most practical cases it is possible to control the temperature and not the energy of the system. Canonical ensemble theory depicts the situation where one has a system with a fixed number of particles, but can exchange energy with the environment with which it is attached. More precisely, a system is kept at a fixed temperature when it is in thermal equilibrium with a much larger system, referred to as a heat bath. Clearly, the energy of the system under study is not a constant, but it has a well defined average energy and the magnitude of the heat energy exchanged with the bath is always small compared to the average energy. We thus have to determine probabilities for states with different energies. The relative recurrence rate of states with different energies is given by their Boltzmann factor, namely, by the quantity e^{-E/k_BT}, where E is the energy of the small system, and k_B is the Boltzmann constant. In the canonical ensemble the system is assigned a temperature, and any state can appear in it. With the small description, we define an all important concept in the canonical ensemble, an ensem-

ble averaged quantity known as the partition function

$$\mathcal{Z} = \sum_{\text{all microscopic states}} e^{-\beta E(\text{microscopic state})} \ . \quad (1)$$

Partition function as such does not have a physical meaning, but this eventually clarifies the connection with thermodynamics. Once we obtain the partition function, then it is easy to derive the *Free energy* which is given by

$$F = -k_B T \ln \mathcal{Z} \ , \quad (2)$$

From this formula we can derive all the thermodynamic properties, also the *Internal energy* (or average energy) can be written as

$$U = -\frac{\partial}{\partial \beta} \ln \mathcal{Z} \ , \quad (3)$$

The *Entropy*

$$S = k_B \{ \ln \mathcal{Z} - \beta \frac{\partial}{\partial \beta} \ln \mathcal{Z} \} \ , \quad (4)$$

and finally *Specific heat*

$$C_V = -k_B (\beta)^2 \frac{\partial U}{\partial \beta} \ . \quad (5)$$

Or

$$C_V = -\beta \frac{\partial S}{\partial \beta} \ . \quad (6)$$

Now we have the required formulae for the calculation of the thermodynamic quantities of the systems which we are interested in.

A. Simple (Bosonic) Harmonic oscillator

Let us briefly describe the well known quantum harmonic oscillator. We have a particle of mass m, trapped in a parabolic well and the entire arrangement vibrates with a natural frequency ω. The whole system is in equilibrium with a thermal reservoir at temperature T. The harmonic oscillator is a particular case in which the degree of freedom takes discrete values, but the number of different values are infinite. The states of the harmonic oscillator are all non-degenerate. Probability of a state with energy E_n is same as the probability of the oscillator having energy E_n, with n as the oscillator quantum number. The probability can be defined as

$$P(E_n) = \frac{1}{\mathcal{Z}} \exp(-\beta E_n) \ , \quad (7)$$

where the partition function can be calculated as

$$\mathcal{Z} = Tr(e^{-\beta \mathcal{H}_B}) \ , \quad (8)$$

Where $\mathcal{H}_B$ denotes the Hamiltonian for the bosonic harmonic oscillator (The subscript B is for bosonic). We can write down the Hamiltonian of the bosonic harmonic oscillator in the (x, p) space and is given by

$$\mathcal{H}_B = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2 \ . \quad (9)$$

It is also possible to write down the Hamiltonian in terms of the creation and annihilation operators $a_B^\dagger$ and a_B (Mathematical details can be found in some standard texbooks like Quantum Mechanics by Zettili [20]) which is given by

$$\mathcal{H}_B = \hbar \omega (a_B^\dagger a_B + \frac{1}{2}) = \hbar \omega (N_B + \frac{1}{2}) \ , \quad (10)$$

The subscript B is to identify the operators as bosonic. We know $a_B^\dagger |n_B\rangle = \sqrt{n_B + 1} |n_B + 1\rangle$ and $a_B |n_B\rangle = \sqrt{n_B} |n_B - 1\rangle$, where the wave function can be written as
$|n_B\rangle = \frac{1}{\sqrt{n_B!}} (a_B^\dagger)^{n_B} |0\rangle_B$. Also $a_B^\dagger a_B = N_B$, where N_B is the number operator. The Bosonic operators obey the commutation relation $[a_B, a_B^\dagger]_- = 1$. We know that $a_B |0\rangle = 0$. The Hamiltonian operating on the state $|n_B\rangle$ can be written as $\mathcal{H}_B |n_B\rangle = (n_B + \frac{1}{2}) \hbar \omega |n_B\rangle$. It is observed that the energy-eigen values are $E_n = (n_B + \frac{1}{2}) \hbar \omega$. Note here that the energy eigenvalues E_n are positive, just as for a classical harmonic oscillator. But in the quantum case, the energies of the oscillator are quantized in intervals of $\hbar \omega$, starting from a non-zero value $E = \frac{1}{2} \hbar \omega$, the *zero point energy*.

With all these requirements, the partition function can be calculated as (cf., Eq.(8))

$$\begin{aligned} \mathcal{Z} &= Tr(e^{-\beta \mathcal{H}_B}) \\ &= \sum_{n_B=0}^{n_B=\infty} \exp(-\beta \hbar \omega (n_B + \frac{1}{2})) \\ &= \frac{1}{2 \sinh(\frac{\beta \hbar \omega}{2})} \ . \end{aligned} \quad (11)$$

From the partition function we can derive all the thermodynamic quantities, and are given by:

Free energy

$$F = \frac{\hbar \omega}{2} + k_B T \ln(1 - e^{-\beta \hbar \omega}) \ , \quad (12)$$

Internal energy

$$U = \frac{\hbar \omega}{2} + \frac{\hbar \omega}{e^{\beta \hbar \omega} - 1} = \frac{\hbar \omega}{2} \coth(\frac{\beta \hbar \omega}{2}) \ , \quad (13)$$

Entropy

$$\frac{S}{k_B} = \frac{\beta \hbar \omega}{e^{\beta \hbar \omega} - 1} - \ln(1 - e^{-\beta \hbar \omega}) \ , \quad (14)$$

alternatively,

$$\frac{S}{k_B} = \frac{\beta \hbar \omega}{2} \coth(\frac{\beta \hbar \omega}{2}) - \ln[2 \sinh(\frac{\beta \hbar \omega}{2})] \ , \quad (15)$$

and finally the *specific heat*

$$\frac{C_V}{k_B} = (\beta\hbar\omega)^2 \mathrm{cosech}^2(\frac{\beta\hbar\omega}{2}) = (\beta\hbar\omega)^2 \frac{e^{\beta\hbar\omega}}{(e^{\beta\hbar\omega}-1)^2} . \quad (16)$$

Thus we have derived all the thermodynamic attributes of a simple bosonic harmonic oscillator.

B. Fermionic Harmonic Oscillator

We have described the bosonic harmonic oscillator in one dimension with a natural frequency ω. The Hamiltonian of such an oscillator can be written in terms of the creation ($a_B^\dagger$) and annihilation (a_B) operator that obey a commutation relation. The Hamiltonian has a symmetric form since we are dealing with Bose particles, so the states must also have a symmetric form. The operators corresponding to bosons obey commutaion relation, and that for fermions satisfy anti-commutation relations. That means the fermionic system have an inherent anti-symmetry associated with it. Therfore let us try out a Hamiltonian for the fermionic harmonic oscillator with a frequency ω of the form [21]

$$\mathcal{H}_F = \frac{\hbar\omega}{2}(a_F^\dagger a_F - a_F a_F^\dagger) . \quad (17)$$

The subscript F is for fermionic. Now, it is interesting to see that, the fermionic Hamiltonian $\mathcal{H}_F = -\frac{\hbar\omega}{2}$, if we assume the operators a_F and $a_F^\dagger$ obey the same commutation relations $[a_F, a_F^\dagger]_- = 1$ as bosons. Then there is no dynamics associated with it. So let us assume that the operators satisfy anti-commutation relations,

$$[a_F, a_F]_+ = 0 = [a_F^\dagger, a_F^\dagger]_+ ,$$

$$[a_F, a_F^\dagger]_+ = 1 = [a_F^\dagger, a_F]_+ . \quad (18)$$

Anti commutators are by definition symmetric. It is quite obvious from the anti-commutation relations that, in such systems the particles must obey Fermi - Dirac statistics. The justification is as follows. If we identify a_F and $a_F^\dagger$ as annihilation and creation operators for such a system, then immediately we can define a number operator

$$N_F = a_F^\dagger a_F , \quad (19)$$

then it is easy to see, $N_F^2 = N_F$ or $N_F(N_F - 1) = 0$. Therefore, the eigenvalues of the number operator can only be zero or one. This is the reflection of the Pauli principle or the Fermi-Dirac statistics. Also it proves that the anti-commutation relations are the natural choice for a fermionic system. We also have

$$[a_F, N_F]_+ = a_F , \qquad [a_F^\dagger, N_F]_+ = -a_F^\dagger . \quad (20)$$

Now the Hamiltonian can be written as

$$\mathcal{H}_F = \hbar\omega(a_F^\dagger a_F - \frac{1}{2}) = \hbar\omega(N_F - \frac{1}{2}) . \quad (21)$$

Now if we assume $|n_F\rangle$ as an eigenstate of N_F, then we have $N_F|n_F\rangle = n_F|n_F\rangle, \;\; n_F = 0, 1$. Let the ground state with no quantum be denoted by $|0\rangle$ and it satisfies

$$N_F|0\rangle = 0 \;\; \text{then} \;\; \Rightarrow \mathcal{H}_F|0\rangle = -\frac{\omega}{2}|0\rangle , \quad (22)$$

simiarly, the state with one quantum is identified to be $|1\rangle$ and it satisfies

$$N_F|1\rangle = |1\rangle \;\; \text{then} \;\; \Rightarrow \mathcal{H}_F|1\rangle = \frac{\omega}{2}|1\rangle . \quad (23)$$

we have $a_F|0\rangle = 0$, $a_F^\dagger|0\rangle = |1\rangle$, $a_F^\dagger|1\rangle = 0$. Hence, in this case the Hilbert space is two dimensional. Also, one more interesting point is that, for a fermionic harmonic oscillator, the ground state energy has the opposite sign from that of a bosonic harmonic oscillator. It is possible to write the Hamiltonian of a bosonic harmonic oscillator in the position and momentum coordinates, which basically equals its classical counterpart. But fermions have no classical analogue, and as a result we cannot directly write down a Lagrangian for the fermionic oscillator with the notions of coordinates and momenta. In order to do that, we need the notion of classical anti-commuting variables. But such variables have been extensively studied in Mathematics under the name of Grasman variables [22]. We are not interested in explaining the details of the Grasman variables here.

As we have identified, the Hamiltonian of a fermionic oscillator given by Eq.(21). The fermionic oscillator is like a two level system with energy eigenvalues

$$E_0 = -\frac{\hbar\omega}{2} , \quad E_1 = \frac{\hbar\omega}{2} . \quad (24)$$

Now the partiton function is trivial and is given by

$$\mathcal{Z} = Tr e^{-\beta\mathcal{H}_F} = 2\cosh(\frac{\beta\hbar\omega}{2}) . \quad (25)$$

So forth the thermodynamic quantities:

Free energy

$$F = -\frac{1}{\beta}(\ln 2\cosh(\frac{\beta\hbar\omega}{2})) , \quad (26)$$

Internal energy (or average energy)

$$U = -\frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{\beta\hbar\omega}+1} = -\frac{\hbar\omega}{2}\tanh(\frac{\beta\hbar\omega}{2}) , \quad (27)$$

Entropy

$$\frac{S}{k_B} = \ln 2\cosh(\frac{\beta\hbar\omega}{2}) - (\frac{\beta\hbar\omega}{2})\tanh(\frac{\beta\hbar\omega}{2}) , \quad (28)$$

alternatively,

$$\frac{S}{k_B} = \ln(e^{\beta\hbar\omega} + 1) - \beta\hbar\omega \frac{e^{\beta\hbar\omega}}{e^{\beta\hbar\omega} + 1} \ , \quad (29)$$

and finally the *specific heat*

$$\frac{C_V}{k_B} = (\beta\hbar\omega)^2 \mathrm{sech}^2(\frac{\beta\hbar\omega}{2}) = (\beta\hbar\omega)^2 \frac{e^{\beta\hbar\omega}}{(e^{\beta\hbar\omega} + 1)^2} \ . \quad (30)$$

The difference in specific heat expressions of the bosonic and the fermionic oscillator is just a matter of sign in the denominator.

C. Supersymmetric Harmonic Oscillator

Supersymmetric oscillator is a simple toy model in quantum field theory, and it is a combination of a bosonic and a fermionic oscillator with the same natural frequency ω [23]. Or in other words, it is a system where one boson and one fermion get trapped in a harmonic well and the whole arrangement vibrates with a frequency ω. The energy eigenstates correspond to a boson with indefinite energy eigenstates and a fermion with two (either zero or one) energy eigenstates. The Hamiltonian is given by

$$\mathcal{H}_S = \mathcal{H}_B + \mathcal{H}_F = \hbar\omega(a_B^\dagger a_B + a_F^\dagger a_F) \ . \quad (31)$$

with $N_B = a_B^\dagger a_B$ and $N_F = a_F^\dagger a_F$. Now Eq.(31) can be expressed as

$$\mathcal{H}_S = \hbar\omega(N_B + N_F) \ . \quad (32)$$

It is clear from the above expression that the energy eigen states of the system will be the eigenstate of the number operators N_B and N_F. Consequently, let us define

$$|N_B, N_F\rangle = |N_B\rangle \otimes |N_F\rangle \ , \quad (33)$$

where

$$\begin{aligned} N_B|n_B\rangle &= n_B|n_B\rangle \ , \quad n_B = 0, 1, 2, ... \\ N_F|n_F\rangle &= n_F|n_F\rangle \ , \quad n_F = 0, 1 \ . \end{aligned} \quad (34)$$

Hence we may write

$$\begin{aligned} \mathcal{H}_S|n_B, n_F\rangle &= E_{n_B,n_F}|n_B, n_F\rangle \\ &= \hbar\omega(n_B + n_F)|n_B, n_F\rangle \ . \end{aligned} \quad (35)$$

So the supersymmetric harmonic oscillator has energy levels $E_n = \hbar\omega n, \ n = n_B + n_F$. This resembles an ordinary harmonic oscillator, but without ground state energy. It is very much clear that the ground state energy of the supersymmetric oscillator vanishes, ie.,

$$E_{0,0} = 0 \ , \quad (a_B|0\rangle = a_F|0\rangle = 0) \ . \quad (36)$$

In the realm of supersymmetry theories, vanishing of the ground state is not an unusual phenomena, it is a consequence of the supersymmetry of the system. From Eq.(35), except for the ground state, all other energy eigenstates are doubly degenerate. Precisely, the states $|n_B, 1\rangle$ and $|n_B + 1, 0\rangle$ have the same energy for any value of n_B. Or, there is one state with $n_F = 0$ and $n_B = n$ and another one with $n_F = 1$ and $n_B = n - 1$, the same as an ordinary harmonic oscillator with ground state energy subtracted. So except for the vaccum state all states come in pairs of one fermionic and one bosonic state with same energy. Admitting to the supersymmetry theory, this degeneracy is also a consequence of the supersymmetry of the system. At this point we will discuss a little bit about the supersymmetry of the system. In the theory, we have two more operators (fermionic) to describe (generally these operators are known as the supercharges):

$$Q_1 = a_B^\dagger a_F \quad \text{and} \quad Q_2 = a_F^\dagger a_B \ . \quad (37)$$

Since the bosonic operators commute with the fermions

$$[Q_1, \mathcal{H}_S]_- = 0 \quad \text{and} \quad [Q_2, \mathcal{H}_S]_- = 0 \ . \quad (38)$$

Therfore Q_1 and Q_2 define conserved quantities of the system and would correspond to the generators of symmetries in the theory. Supercharges act on the paired states by transforming a bosonic state to a fermionic state and vice versa. The only exception is the ground state which is annihilated by the supercharges. This is a typical situation for the supersymmetric system. There exists models in which one has a supersymmetry algebra, but ground state has nonzero energy. In the language of field theory, the ground state is not supersymmetric, therfore the model is not supersymmetric, supersymmetry is spontaneously broken. The anti-commutator of the supercharges is given by

$$[Q_1, Q_2]_+ = \frac{1}{\hbar\omega}\mathcal{H}_S \ . \quad (39)$$

Certainly, Eqs (37), (38) and (39) indicates that, Q_1, Q_2 and $\mathcal{H}_S$ constitute an algebra which involves both commutators and anti commutators. This algebra is known as the *graded Lie algebra*. This algebra have a beautiful consequence, namely, if the ground state is invariant under the supersymmetry transformations (ie., $Q_1|0\rangle = 0 = Q_2|0\rangle$ and $\langle 0|\mathcal{H}_S|0\rangle = \hbar\omega\langle 0|Q_1Q_2 + Q_2Q_1|0\rangle = 0$), then the ground state vanishes in supersymmetry theory. Q_2 is really a hermitian conjugate of Q_1.

Energy eigenvalues of the supersymmetric oscillator is

$$E_n = \hbar\omega n, \ n = n_B + n_F \ . \quad (40)$$

It is now straight forward to calculate the canonical par-

tition function:

$$\begin{aligned}\mathcal{Z} &= \sum_{n=0}^{n=\infty} e^{-\beta\hbar n} \\ &= 1 + 2\sum_{n=1}^{n=\infty} e^{-\beta\hbar n} \\ &= \frac{e^{\beta\hbar\omega}+1}{e^{\beta\hbar\omega}-1} \ . \end{aligned} \tag{41}$$

Or

$$\mathcal{Z} = \coth(\frac{\beta\hbar\omega}{2}) \ . \tag{42}$$

Now as usual the thermodynamic quantities follows:

Free energy

$$F = -\frac{1}{\beta}\ln\coth(\frac{\beta\hbar\omega}{2}) \ , \tag{43}$$

Internal energy (or average energy)

$$U = \frac{\hbar\omega}{\sinh(\beta\hbar\omega)} \ , \tag{44}$$

Entropy

$$\frac{S}{k_B} = \ln\coth(\frac{\beta\hbar\omega}{2}) + (\beta\hbar\omega)\mathrm{cosech}(\beta\hbar\omega) \ , \tag{45}$$

and finally the *specific heat*

$$\frac{C_V}{k_B} = (\beta\hbar\omega)^2\mathrm{cosech}(\beta\hbar\omega)\coth(\beta\hbar\omega) \ . \tag{46}$$

In an alternative form, specific heat can be written as

$$\frac{C_V}{k_B} = 2(\beta\hbar\omega)^2\frac{e^{\beta\hbar\omega}}{(e^{2\beta\hbar\omega}-1)}(\frac{e^{2\beta\hbar\omega}+1}{e^{2\beta\hbar\omega}-1}) \ . \tag{47}$$

This expression can be represented in terms of the bosonic specific heat as

$$(\frac{C_V}{k_B})_S = 2(\frac{C_V}{k_B})_B(e^{2\beta\hbar\omega}+1) \ . \tag{48}$$

The specific heat has more bosonic character than fermionic. Hence the variation of the specific heat with temperature is expected to be more bosonic in nature.

III. HARMONICALLY BOUND ELECTRON IN A MAGNETIC FIELD

The problem of an electron in a uniform and homogeneous magnetic field attained tremendous attention nowadays in mesoscopic physics [24] and quantum dots or the quantum confined nanostructures[25]. It was Landau who emulated this problem and conveyed that the problem of an elctron in a magnetic field is essentially quantum mechanical and can be solved exactly through the first principles of quantum mechanics. We are interested in the thermodynamics of this problem with an additional harmonic oscillator potential.

Since the energy eigen spectrum of the free electron under the magnetic field (both in the absence and presence of the parabolic confinement) shows simple harmonic oscillator structure, we must commit to say that the quantum mechanical formulation is identical to that of a harmonic oscillator. Moreover, harmonic oscillator potential is the only case where simple results can be obtained. It allows one to discuss in detail the effects of the boundary and the way in which the bulk limit is attained. This is the reason we are interested in deriving the thermodynamic properties of this Landau-Fock-Darwin model (LFD). It is worth to study the effect of spin of the electron trapped, on the model, so we analyse two cases viz., the LFD model with and without spin. For the sake of simplicity we have considered the case where the electron possesses spin $\frac{1}{2}$.

A. Without Spin

Here, the magnetic behavior of the electron has been considered. Permanent magnetic dipole moment of electron is ignored and only the influence of the external magnetic field on the electron is taken into account. The Hamiltonian for a non-relativistic electron of mass m and charge e confined in a two dimensional isotropic harmonic bowl (parabolic well) of frequency ω_0 can be written as

$$\mathcal{H}_0 = \frac{1}{2m}(\mathbf{P} - \frac{e}{c}\mathbf{A})^2 + \frac{1}{2}m\omega_0^2\mathbf{r}^2 \ . \tag{49}$$

Where $\mathbf{P}$ and $\mathbf{r}$ are two-dimensional vectors and $\mathbf{A}$ is the magnetic vector potential. Using the "symmetric gauge" $\mathbf{A} = (Hy, -Hx, 0)$, we can write the Hamiltonian in Eq.(49) as

$$\begin{aligned}\mathcal{H}_0 &= \frac{1}{2m}[(p_x - \frac{eyH}{2c})^2 + (p_y + \frac{exH}{2c})^2] \\ &+ \frac{1}{2}m\omega_0^2(x^2+y^2) \ . \end{aligned} \tag{50}$$

The energy eigen values of this particular problem is known as the Fock-Darwin spectrum which basically has a form

$$E_{n_1,n_2} = \hbar\sqrt{\omega_0^2 + (\frac{\omega_c}{2})^2}(n_1+n_2+1) + \frac{1}{2}\hbar\omega_c(n_1-n_2) \ . \tag{51}$$

Where $\omega_c = \frac{eB}{mc}$, the cyclotron frequency, c is the velocity of light. The Fock-Darwin energy spectrum is closely related to the energy eigen spectrum of an ideal rotating BEC, with the trap is circular in cross section. This spectrum is interesting because it shows how smoothly the spectrum interpolates between the bound states of a two-dimensional harmonic oscillator (when $\omega_c = 0$) to the case of the Landau levels of a free particle in a magnetic field (when $\omega_0 = 0$). The Hamiltonian (cf., Eq.

(50)) can be effectively canonically transformed into the sum of two independent one dimensinal harmonic oscillators with two frequencies ω_+ and ω_- either by introducing the generating function technique of Valatin[26] or by quantum mechanical operators[27]. We define two operators

$$a_{\pm} = \frac{1}{2l}[x \mp iy] - \frac{l}{2}[\partial_x \mp i\partial_y] \ , \tag{52}$$

with $l = \sqrt{\frac{\hbar}{m\omega}}$ is the Fock-Darwin length. Now it is easy to write the Hamiltonian in terms of these operators and it turns out to be

$$\mathcal{H}_0 = (a_+^\dagger a_+ + \frac{1}{2})\hbar\omega_+ + (a_-^\dagger a_- + \frac{1}{2})\hbar\omega_- \ , \tag{53}$$

where

$$\begin{aligned} \omega_+ &= \omega + \frac{\omega_c}{2} \ , \\ \omega_- &= \omega - \frac{\omega_c}{2} \ , \end{aligned} \tag{54}$$

with $\omega = \sqrt{\omega_0^2 + \frac{\omega_c^2}{4}}$. Now identifying $n_1 = a_+^\dagger a_+$ and $n_2 = a_-^\dagger a_-$ gives rise to the energy eigen values

$$E_{n_1,n_2} = (n_1 + \frac{1}{2})\hbar\omega_+ + (n_2 + \frac{1}{2})\hbar\omega_- \ , \tag{55}$$

which is the same as Eq.(51). In the absence of a magnetic field the Fock-Darwin levels are degenerate, $\omega_+ = \omega_- = \omega_0$, whereas a strong magnetic field leads to the formation of the structure of Landau energy levels, represented by the cyclotron energy $\hbar\omega_+ \approx \hbar\omega_c$.

Thermodynamics of this problem has been studied by Jishad et al [19], where they have used the path integral approach to calculate the partition function. We obtain the partition function in the canonical ensemble as

$$\mathcal{Z} = \mathcal{Z}_+\mathcal{Z}_- = \frac{1}{2\sinh(\frac{\beta\hbar\omega_+}{2})}.\frac{1}{2\sinh(\frac{\beta\hbar\omega_-}{2})} \ . \tag{56}$$

Now the thermodynamic quantities: *Free energy*

$$F = \frac{1}{\beta}(\ln(2\sinh(\frac{\beta\hbar\omega_+}{2})) + \ln(2\sinh(\frac{\beta\hbar\omega_-}{2}))) \ , \tag{57}$$

the *internal energy* is given by

$$U = \frac{\hbar\omega_+}{2}\coth\left(\frac{\beta\hbar\omega_+}{2}\right) + \frac{\hbar\omega_-}{2}\coth\left(\frac{\beta\hbar\omega_-}{2}\right) \ . \tag{58}$$

Entropy

$$\begin{aligned} \frac{S}{k_B} &= -\ln(2\sinh(\frac{\beta\hbar\omega_+}{2})) - \ln(2\sinh(\frac{\beta\hbar\omega_-}{2})) \\ &+ \frac{\beta\hbar\omega_+}{2}\coth(\frac{\beta\hbar\omega_+}{2}) + \frac{\beta\hbar\omega_-}{2}\coth(\frac{\beta\hbar\omega_-}{2}) \ , \end{aligned} \tag{59}$$

finally *Specific heat*

$$\begin{aligned} \frac{C_V}{k_B} &= \left(\frac{\beta\hbar\omega_+}{2}\right)^2 \mathrm{cosech}^2(\frac{\beta\hbar\omega_+}{2}) \\ &+ \left(\frac{\beta\hbar\omega_-}{2}\right)^2 \mathrm{cosech}^2(\frac{\beta\hbar\omega_-}{2}) \ . \end{aligned} \tag{60}$$

In the limit of the vanishing magnetic field ($\omega_c = 0$), this problem reduces to the problem of two isotropic harmonic oscillator of same frequency. Also in the limit of vanishing oscillator potential ($\omega_0 \to 0$), we can recover the the thermodynamics of the original Landau problem (ie., the electron in a magnetic field). In the vanishing oscillator potential limit (ie., $\omega_0 \to 0$ or $\omega_- \to 0$) we cannot recover the partition function of the original Landau problem form Eq.(56) because Eq.(56) diverges in this limit, since the system becomes translationally invariant in space. Recovery of partition function for Landau problem is well expalined in [19]. Apart from the partition function none of the thermodynamic quantities are plagued by this issue.

B. With Spin

Assume that the electron possesses an intrinsic spin of value $\frac{1}{2}\hbar\hat{\sigma}$ and a magnetic moment μ_B, where $\hat{\sigma}$ is the Pauli spin operator and $\mu_B = \frac{e\hbar}{2mc}$. Here the spin of the electron can have two possible orientations one is $\uparrow$ or $\downarrow$ with respect to the applied magnetic field $\vec{\mathbf{H}}$. We assume the magnetic field is in the z direction, ie., $\vec{\mathbf{H}} = H_z$. The Hamiltonian with the inclusion of spin can be obtained by adding $-\mu_B(\hat{\sigma}.H)$ to Eq.(49). The total Hamiltonian is given by

$$\mathcal{H} = \frac{1}{2m}(\mathbf{P} - \frac{e}{c}\mathbf{A})^2 + \frac{1}{2}m\omega_0^2\mathbf{r}^2 - \mu_B(\hat{\sigma}.H) \ . \tag{61}$$

The first term in the Hamiltonian gives rise to diamagnetism and the last term is responsible for the paramagnetism. With Eq.(61), now the energy eigenvalues can be written as $E_{n_1,n_2,\sigma} = E_{n_1,n_2} - \mu_B\sigma_z H$, Hence the total partition function is given by

$$\mathcal{Z} = \mathcal{Z}_+(\beta,\omega_+)\mathcal{Z}_-(\beta,\omega_-)\mathcal{Z}_s(\beta) \ , \tag{62}$$

where $\mathcal{Z}_s(\beta)$ is the factor introduced by the spin:

$$\begin{aligned} \mathcal{Z}_s(\beta) &= Tr e^{-\beta(-\mu_B H\hat{\sigma}_z)} \\ &= Tr\begin{pmatrix} e^{\beta\mu_B H} & 0 \\ 0 & e^{-\beta\mu_B H} \end{pmatrix} \\ &= 2\cosh(\beta\mu_B H) \ . \end{aligned} \tag{63}$$

Hence the total partition function turns out to be

$$\mathcal{Z} = \frac{2\cosh(\frac{\beta\hbar\omega_c}{2})}{2\sinh(\frac{\beta\hbar\omega_+}{2}).2\sinh(\frac{\beta\hbar\omega_-}{2})} \ . \tag{64}$$

The thermodynamic quantities: *Free energy*

$$F = \frac{1}{\beta}\{\ln(2\sinh(\frac{\beta\hbar\omega_+}{2})) + \ln(2\sinh(\frac{\beta\hbar\omega_-}{2})) - \ln(2\cosh(\frac{\beta\hbar\omega_c}{2}))\}\ , \quad (65)$$

the *internal energy*

$$U = \frac{\hbar\omega_+}{2}\coth(\frac{\beta\hbar\omega_+}{2}) + \frac{\hbar\omega_-}{2}\coth(\frac{\beta\hbar\omega_-}{2}) - \frac{\hbar\omega_c}{2}\tanh(\frac{\beta\hbar\omega_c}{2})\ . \quad (66)$$

Also *entropy* can be calculated as

$$\frac{S}{k_B} = -\ln(2\sinh(\frac{\beta\hbar\omega_+}{2})) - \ln(2\sinh(\frac{\beta\hbar\omega_-}{2})) + \ln(2\cosh(\frac{\beta\hbar\omega_c}{2})) + \frac{\beta\hbar\omega_+}{2}\coth(\frac{\beta\hbar\omega_+}{2}) + \frac{\beta\hbar\omega_-}{2}\coth(\frac{\beta\hbar\omega_-}{2}) - \frac{\beta\hbar\omega_c}{2}\tanh(\frac{\beta\hbar\omega_c}{2})\ , \quad (67)$$

and the *specific heat*

$$\frac{C_V}{k_B} = (\frac{\beta\hbar\omega_+}{2})^2\mathrm{cosech}^2(\frac{\beta\hbar\omega_+}{2}) + (\frac{\beta\hbar\omega_-}{2})^2\mathrm{cosech}^2(\frac{\beta\hbar\omega_-}{2}) - (\frac{\beta\hbar\omega_c}{2})^2\mathrm{sech}^2(\frac{\beta\hbar\omega_c}{2})\ . \quad (68)$$

Again in the limit of vanishing magnetic field, the resuts for a free electron (in two-dimensions) in a magnetic field is recovered.

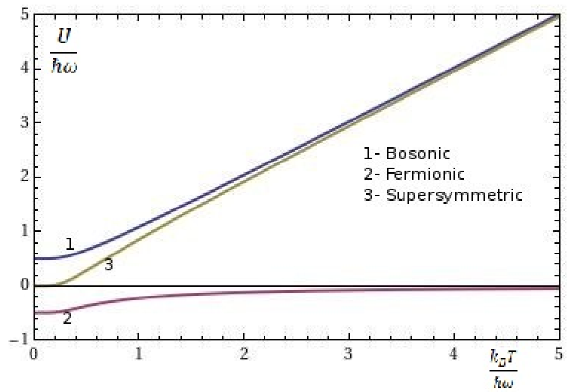

FIG. 1: The variation of internal energy with respect to the temperature, for each oscillator.

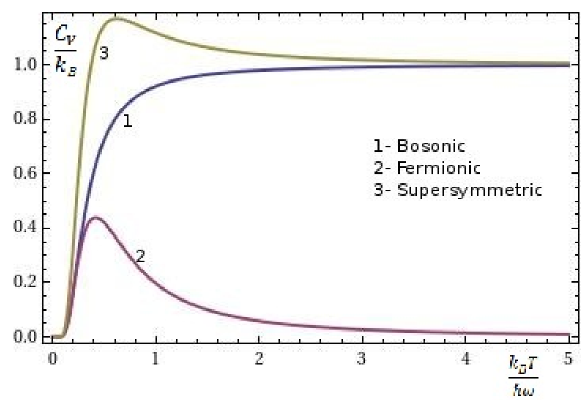

FIG. 2: The variation of specific heat with respect to the temperature

IV. RESULTS AND DISCUSSION

A. Bosonic, Fermionic and Supersymmetric Oscillators

Bosonic oscillator: The behavior of thermodynamic properties of a simple bosonic harmonic oscillator both at low and high temperatures, is well studied. For a comprehensive explanation we discuss these here. As $T \to \infty$ (ie., $\beta\hbar\omega \to 0$), the average energy $U \approx k_BT + ...$, classical value is exactly recovered. In this problem we have a characteristic parameter $\frac{\hbar\omega}{k_BT}$, which is the ratio of the energy levels of the oscillator to the mean thermal energy available. If k_BT is very large, discrete structure of the energy spectrum does not show up, one would obtain the classical value. In this limit, the specific heat converges, as expected. On the other hand, in the limit of $T \to 0$, we may see the largest deviations from the classical case in the internal energy and it turns out to be $U \approx \frac{\hbar\omega}{2}$. Apparently, the specific heat vanishes like $x^2e^{-x}(x \to \infty)$.

The vanishing of specific heat at low temperatures is a characteristic feature of all quantum systems with a discrete energy spectrum. One way of explaining this effect is: if the thermal energy k_BT is much smaller than the oscillator energy $\hbar\omega$, then an oscillator cannot(only with extremely small probability)be excited to the higher energy level. Now the system is absolutely incapable of absorbing energy provided by the heat bath. But classical systems can absorb arbitrarily small energies k_BT. Variation of entropy is as expected. At low temperatures, the entropy approaches zero and at high temperatures entropy is infinite.

Fermionic Oscillator: At low temperatures, the internal energy of the system is like $U \approx \frac{-\hbar\omega}{2}$, ie., the system tries to remain in the ground state. At high temperatures, the variation of internal energy with respect to temperature is $U \approx \frac{-\hbar^2\omega^2}{4k_BT}$, means the syetm tries to populate the two available energy states. The specific heat of the system is vanishingly small at low temperatures, and that vary in a form which is given by $\frac{C_v}{k_B} \approx (\frac{\hbar\omega}{k_BT})^2e^{-\frac{\hbar\omega}{k_BT}}$. The specific heat vanishes at high temperature as well in a way $\frac{C_v}{k_B} \approx (\frac{\hbar\omega}{2k_BT})^2$. Somewhere around $T = \frac{\epsilon}{k_B}$, it displays a maximum. Writing Δ for the energy difference between the two allowed states of the system, we may

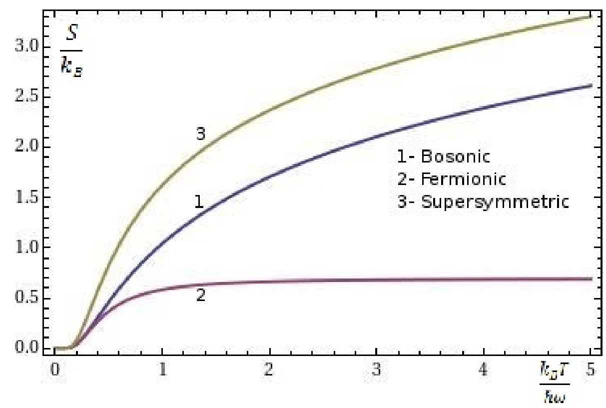

FIG. 3: The variation of entropy with temperature

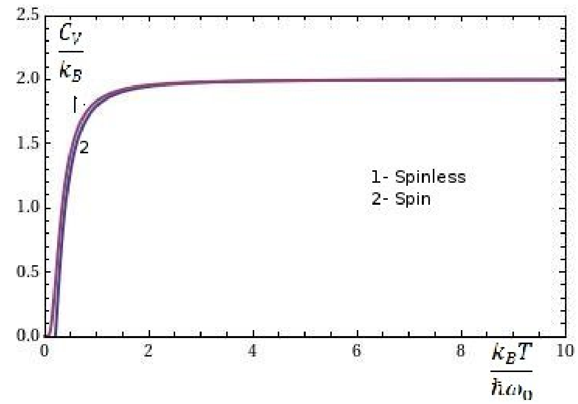

FIG. 4: The variation of specific heat when $\omega_c = 0.4\omega_0$, with temperature

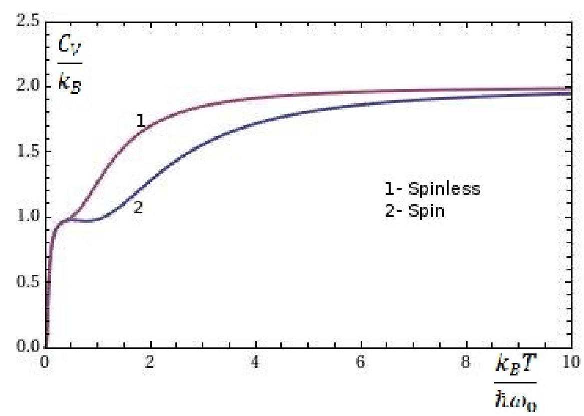

FIG. 5: The variation of specific heat when $\omega_c = 4\omega_0$, with temperature

write the specific heat as

$$\frac{C_v}{k_B} = (\frac{\Delta}{k_B T})^2 e^{\frac{\Delta}{k_B T}} (1 + e^{\frac{\Delta}{k_B T}})^{-2} \ . \tag{69}$$

A specific heat of this form is generally known as the *Schottky specific heat*; characterized by an anomalous peak, and it is observed in all systems with an excitation gap Δ. The variation of entropy is as expected, it is vanishingly small at low temperaures ($k_B T << \hbar\omega$); it rises to a maximum rapidly when $k_B T$ is of the order of the energy difference between the two states and it approaches a limiting value $k_B \ln 2$ at $k_B T >> \hbar\omega$.

Supersymmetric oscillator: The variation of the internal energy of a supersymmetric oscillator is almost like that of a bosonic oscillator. Here the specific heat is also vanishing exponentially at low temperatures just as in the case of a bosonic oscillator. Here also we can see at some temperature T, the specific heat displays a maximum. The specific heat of the supersymmetric oscillator vanishes exponentially at low temperatures as $\frac{C_V}{k_B} \approx 2(\frac{\hbar\omega}{k_B T})^2 e^{-\frac{\hbar\omega}{k_B T}}$, ie., twice as that of a fermionic (or bosonic) oscillator at low-temperatures. At high temperatures, the variation of specific heat is like $(1 + \frac{\hbar\omega}{k_B T})^2$, which is the square of the respective quantity of a bosonic oscillator at high temperatures. The entropy vanishes exponentially ($\approx \beta\hbar\omega e^{-\beta\hbar\omega}$) at very low temperatures, and goes to infinity at very high temperatures.

The variation of internal energy, entropy and specific heat of all the three oscillators with respect to temperature are plotted in Fig.1, Fig.2 and Fig.3.

B. Harmonically bound electron in a magnetic field

We have two temperature scales, one is $k_B T/\hbar\omega_0$, which appears in the oscillator problem and the other is $k_B T/\hbar\omega_c$, set by the magnetic field. It is to be noted here that the arguments of the hyperbolic terms in the expression for the specific heat contains both the temperature scales. We plot the thermodynamic quantities with respect to the temperature scale $k_B T/\hbar\omega_0$. Let us define $\frac{1}{w} = \frac{\omega_c}{2\omega_0}$, and $\tilde{T} = k_B T/\hbar\omega_0$. So that $\hbar\omega_c/2k_B T = \frac{1}{w\tilde{T}}$ and we may write $\frac{\beta\hbar\omega_\pm}{2} = \frac{1}{2\tilde{T}}(\sqrt{1 + (\frac{1}{w})^2} \pm \frac{1}{w})$. If the cyclotron frequency is much greater than the oscillator frequency ie., $\omega_c >> \omega_0$, then $\omega_+ \approx \omega_c$ and $\omega_- \approx 0$. That means a magnetic dominancy is clear. At very low-temperatures, the internal energy goes as $U \approx \hbar\sqrt{\omega_0^2 + \frac{\omega_c^2}{4}}$, and if the magnetic field is high, ie., $\omega_c >> \omega_0$, then it is approximately equal to $\frac{\hbar\omega_c}{2}$, which is similiar to the one dimensional harmonic oscillator low-temperature expression with the oscillator frequency being replaced by the cyclotron frequency. If the magnetic field is very low, ie., $\omega_c << \omega_0$, then $U \approx \hbar\omega_0$, the ground state energy of a 2D isotropic oscillator. In addition to this results if we take the electron spin into account, then in high magnetic fields, internal energy seems to be vanishing!, and produces the value $\hbar\omega_0$ at very low magnetic fields. At high temperatures, the internal energy (both in the absence and presence of spin) varies like $U \approx 2k_B T + ...$ which is the expected classical value. At high temperatures, the specific heat varies with temper-

ature like

$$(\frac{C_V}{k_B})_{spin=0} = 2 - (\frac{\hbar}{k_B T})^2 \frac{(2\omega_0^2 + \omega_c^2)}{6} + O(\frac{1}{T^2}) \ , \quad (70)$$

and similiarly

$$(\frac{C_V}{k_B})_{spin=\frac{1}{2}} = 2 - (\frac{\hbar}{k_B T})^2 \frac{(\omega_0^2 + 2\omega_c^2)}{6} + O(\frac{1}{T^2}) \ . \quad (71)$$

Clearly, it is understood that the classical equipartition theorem is satisfied at very high temperatures, with a degree of freedom of 2 (since this is a two dimensional problem).

At low temperatures the behavior of the specific heat for a spinless electron in the presence of a parabolic potential and magnetic field is just identical to that of two independant isotropic simple harmonic oscillators as it is evident from the Eq.(60). As $T \to 0$, the specific heat vanishes like

$$\frac{C_v}{k_B} \approx [(\beta\hbar\omega_+)(e^{-\frac{\beta\hbar\omega_+}{2}})]^2 + [(\beta\hbar\omega_-)(e^{-\frac{\beta\hbar\omega_-}{2}})]^2 \ . \quad (72)$$

While the heat capacity at low-temperature in the case where the electron possess the spin angular momentum of $\frac{1}{2}$ has an additional term due to the spin contribution. This also goes like $\approx [(\beta\hbar\omega_c)(e^{-\frac{\beta\hbar\omega_c}{2}})]^2$. Hence the specific heat (with spin), decays as

$$\begin{aligned} \frac{C_v}{k_B} \ &\approx \ [(\beta\hbar\omega_+)(e^{-\frac{\beta\hbar\omega_+}{2}})]^2 + [(\beta\hbar\omega_-)(e^{-\frac{\beta\hbar\omega_-}{2}})]^2 \\ &- \ [(\beta\hbar\omega_c)(e^{-\frac{\beta\hbar\omega_c}{2}})]^2 \ . \quad (73) \end{aligned}$$

In the limit $\omega_0 >> \omega_c$ (ie., very low magnetic field strength) both the hybrid frequencies ω_+ and ω_- approximately equal to ω_0, the oscillator characteristic frequency. Apparently, for very low magnetic fields ($\frac{\omega_c}{2\omega_0} \to 0$), the heat capacity for both cases follow the same form upto the leading order, which is given by

$$\frac{C_V}{k_B} \approx 2(\frac{\hbar\omega_0}{2k_B T})^2 \mathrm{cosech}^2(\frac{\hbar\omega_0}{2k_B T}) \ , \quad (74)$$

which is nothing but the specific heat of a simple harmonic oscillator with the factor of 2 due to the extra degree of freedom. As $T \to 0$, the above expression vanishes exponentially as $2(\beta\hbar\omega_0 e^{-\frac{\beta\hbar\omega_0}{2}})^2$. If the spin is present, then the specific heat goes (at low-temperatures) like

$$\frac{C_v}{k_B} \approx 2(\beta\hbar\omega_0 e^{-\frac{\beta\hbar\omega_0}{2}})^2 - (\beta\hbar\omega_c e^{-\frac{\beta\hbar\omega_c}{2}})^2 \quad (75)$$

which shows a clear indication that the harmonic oscillator contribution dominates over the magnetic contribution. This result is expected too. When the magnetic field strength is comparatively small, the contribution from the spin angular momentum of the electron ($\mu_B \sigma_z H$) is nullified by the other part of the energy (which is harmonic in nature) ie., E_{n_1,n_2}. Heat capacity versus temperature is plotted for both low and high

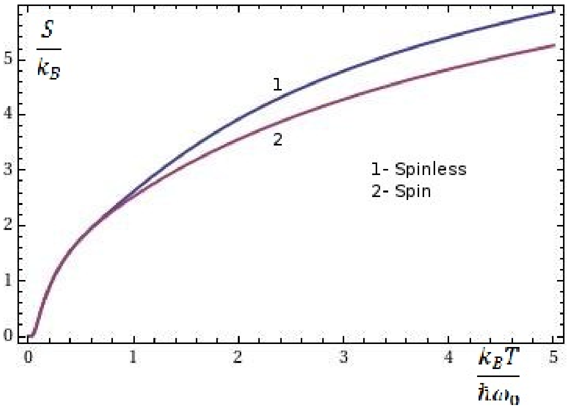

FIG. 6: The variation of entropy when $\omega_c = 4\omega_0$, with temperature

values of magnetic fields, in Fig.4 and Fig.5 respectively. From the Fig.4 it is clear that at very low temperatures and low magnetic field strength (ie., $\omega_c << \omega_0$), the specific heat shows the behavior of an Einstein oscillator, ie., an exponential suppression. Conversely, as the strength of the magnetic field is increased, the amalgamated curves in Fig.4 started bifurcating after a certain value of the temperature. For very high values of the magnetic field, ie., when $\omega_c >> \omega_0$, then $\omega_+ \approx \omega_c$ and $\omega_- \approx 0$. The variation of specific heat at low temperatures and in high magnetic field resembles the varaiation of the specific heat (with temperature) of a harmonic oscillator at low-temperatures, but with the harmonic oscillator frequency being replaced by the cyclotron frequency. ie.,

$$\frac{C_v}{k_B} \approx (\beta\hbar\omega_c e^{-\frac{\beta\hbar\omega_c}{2}})^2 \ . \quad (76)$$

From Eq.(76), it is evident that $\frac{C_v}{k_B} \approx 0$ at high magnetic fields and at low-temperatures, and we can observe a small and sudden drop or dip in specific heat as indicated in Fig.5. But neverthless, when the system is attenuated with the thermal fluctuations, ie., at high temperatures, all contributions are nullified and we can observe a clear resumption of the classical equipartiton value. The entropy variation is as expected. At very low-temperatures, the degree of disorder in the system is negligible and the system is said to be in the ground state. At low-temperatures the contributions from each terms of the Eq.(59) and (67) cancels each other to leave no contribution to the entropy by validating the third law of thermodynamics. We observe from the Figs.6 and 7, that both the curves follow same path at very low-temperatures and get bifurcated and approaches infinity at high temperatures. Variation of entropy with temperature is given in Fig.6 and Fig.7, both for high and low magnetic field strengths.

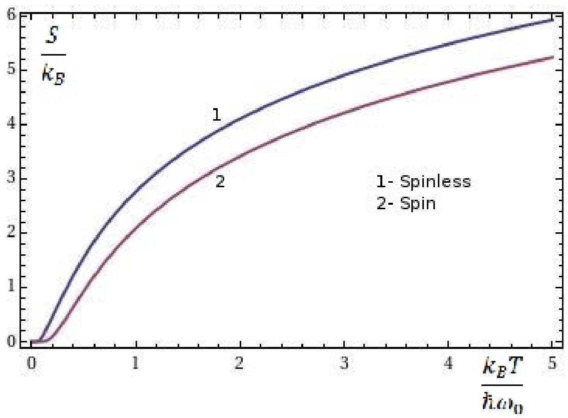

FIG. 7: The variation of entropy when $\omega_c = (0.4)\omega_0$, with temperature

V. CONCLUSION

We have examined the thermodynamics of the toy models of quantum mechanics viz., a bosonic harmonic oscillator, a fermionic oscillator and the combination of the both, a supersymmetric oscillator. We have graphically analysed the variation of the internal energy, specific heat and the entropy of these systems respectively with respect to temperature. All the low-temperature properties are in conformity with the third law of thermodynamics and particulary specific heat showed an exponetial suppression in the low-temperatures. We have done a detailed calculation of the thermodynamic quantities of the well studied Landau-Fock-Darwin model which can be effectively converted into a harmonic oscillator problem, both in the absence and presence of the spin angular momentum of the electron and the graphical results are in much coordination with the analytical results. We have discussed in detail both the low-temperature (both in the low ($\omega_c << \omega_0$) and high ($\omega_c >> \omega_0$) magnetic fields) and high temperature variations of the respective thermodynamic quantities. In view of the nanoscopic or mesoscopic systems, the LFD model has got numerous applications and is worth to study its thermodynamics.

VI. ACKNOWLEDGMENTS

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